

The thermodynamics of urban population flows

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Orderliness, reflected via mathematical laws, is encountered in different frameworks involving social groups. Here we show that a thermodynamics can be constructed that macroscopically describes urban population flows. Microscopic dynamic equations and simulations with random walkers underlie the macroscopic approach. Our results might be regarded, via suitable analogies, as a step towards building an explicit social thermodynamics.

I. INTRODUCTION

The application of mathematical models to social sciences has a long and distinguished history¹. One may speak of empirical data from scientific collaboration networks², cites of physics journals³, the Internet traffic⁴, Linux packages links⁵, popularity of chess openings⁶, as well as electoral results^{7,8}, urban agglomerations^{9,10} and firm sizes all over the world¹¹. A specially relevant issue is that of universality classes defined by the so-called Zipf's law (ZL) in the cumulative distribution or rank-size distributions^{3-6,9-15}. Maillart et al.⁵ have found that links' distributions follow ZL as a consequence of stochastic proportional growth. Such kind of growth assumes that an element of the system becomes enlarged proportionally to its size k , being governed by a Wiener process. The class emerges from a condition of stationarity (dynamic equilibrium)¹⁵. ZL also applies for processes involving either self-similarity⁶ or fractal hierarchy¹³, all of them mere examples amongst very general stochastic ones¹⁴. A second universality-class was found by Costa Filho et al.⁷, who studied vote-distributions in Brazil's electoral results. Therefrom emerge multiplicative processes in complex networks⁸. Such behavior ensues as well in i) city-population rank distributions¹⁶, ii) Spanish electoral results¹⁶, and iii) the degree distribution of social networks¹⁷. As shown in Ref.¹⁸, this universality class encompasses Benford's Law¹⁹. In the present vein, still another kind of idiosyncratic distribution is often reported: the log-normal one²⁰, that has been observed in biology (length and sizes of living tissue²¹), finance (in particular, the Black and Scholes model²²), and firms-sizes. The latter instance obeys Gibrat's rule of proportionate growth²³, that also applies to cities' sizes.

Together with geometric Brownian motion, there is a variety of models arising in different fields that yield Zipf's law and other power laws on a case-by-case basis^{9,10,15,24-26}, as preferential attachment⁴ and competitive cluster growth^{8,17} in complex networks, used to explain many of the scale-free properties of social networks. For instance, we may mention detailed realistic approaches in urban modelling^{27,28}, opinion dynamics²⁹, and electoral results^{7,30}. Of course, the renormalization group is intimately related to scale invariance and asso-

ciated techniques have been fruitfully exploited in these matters (as a small sample see 26,31).

It has been recently shown, in Ref. 32, that a variational principle based on MaxEnt can be successfully applied to scale-invariant social systems. Used in the present context, it allows for a classification of the above cited behaviors on the basis of inferences drawn from objective observables of the system. We had also shown³³ that including some dynamical information in the variational scheme³⁴ one is able to reproduce the shape of empirical city-population distributions, going beyond the customary universality classes conventionally used in such regards. Indeed, a connection between explicit microscopic growth equations and the macroscopic characterization exists, illustrated for logistic-growth in Ref. 35. We will here describe the manner in which the methods of that paper can be generalized to first-principles theoretical framework describing population flows in terms of thermodynamic concepts.

A. Motivation, statement of the problem and goal

We are looking here for more than models: what we aim for is to discover physical principles that may underlie some social phenomena. Our system is a specific geographical area whose population is distributed amongst several population-nuclei (cities, villages, towns, etc.) Each nucleus' population is time-dependent due to migration, birth, death, etc. Our aim is to quantitatively describe the population-nuclei's variation. Microscopic variables are plentiful, but our main goal is to be able to identify macroscopic variables that can give a reasonable account of urban population-variations.

We will proceed in seven steps, as indicated in the scheme below:

1. Introduce the basic observables and the empirical data sets.
2. Identify the stochastic nature of the city-population growth rates.
3. Postulate dynamic microscopic equations and empirically validate them.
4. Perform numerical simulations with random walkers following these dynamical equations and parametrize the macroscopic evolution.
5. Show that equilibrium configurations of such evolutions can be predicted by MaxEnt using few macroscopic parameters.
6. Derive thermodynamic-like relations between these macro-parameters.
7. Show the applicability of our thermodynamic description by modeling empirical urban flows as an scale invariant ideal gas.

The paper is organized as follows. Step 1 is addressed in the next Section II. Section III deals with step 2, Section IV with step 3, Section V with step 4, and Section VI with step 5 and 6. Finally, the application is dealt with in Section VII, and some conclusions are drawn in Section VIII.

II. PRELIMINARY MATTERS

The basic ingredients we need in our approach, following Refs. 33,34, are

- i) n , the total number of “population-nuclei”;
- ii) $x_i(t)$, the population of the i -th nucleus at time t (and $\mathbf{x}(t) = \{x_i(t)\}_{i=1}^n$ a vector with all the populations);
- iii) x_0 and x_M , the minimum and maximum allowed nucleus’ population (in general $x_0 = 1$ and $x_M = \infty$);
- iv) N_T , the total area’s population ($N_T = \sum_{i=1}^n x_i(t)$);
- v) $\dot{x}_i(t)$, the time-derivative of $x_i(t)$ (thus the pairs $\{(x_i, \dot{x}_i)\}_{i=1}^n$ compose the “urban phase space”); and
- v) some a priori knowledge of the dynamics at hand, written as

$$\dot{x}_i(t) = k_i(t)g_i[\mathbf{x}(t)] \quad (1)$$

where g_i are population-functions to be determined and $k_i(t)$ growth rates independent of the g_i .

The raw data used in our analysis is obtained from the Spanish state institute INE³⁶ and cover annually the period 1996-2010 (with the exception of 1997). It encompasses up to 8000 municipalities (the smallest Spanish administrative unit) distributed within 50 provinces (the building blocks of the autonomous communities). We use provinces and municipalities as the closest representatives of the ideal of a closed system’s fundamental elements. Also other regions of the world are used as examples along the text. In this tableau, the total population N_T of a province is apportioned in n nucleus. The i -th nucleus account a population of $x_0 \leq x_i(t) \leq x_M$ at time t , which time-evolution obeys Eq. (1).

III. THE STOCHASTIC NATURE OF POPULATION GROWTH RATES

We begin dealing with step 2 of our Scheme, saying something meaningful concerning the form of the growth rates $k_i(t)$ in Eq. (1). The value of k_i above depends upon millions of individual decisions, so it is expected some stochastic behavior. We should know both the average $m_i = \langle \dot{x}_i(t) \rangle_{\delta t}$ and the standard deviation $s_i = \langle (\dot{x}_i(t) - m_i)^2 \rangle_{\delta t}^{1/2}$ (for each i) in a time-window δt around t . Trying then to study the distribution of $\xi_i(t) = (\dot{x}_i(t) - m_i)/s_i$ one immediately finds

$$\begin{aligned} m_i &= \langle k_i(t)g_i(t) \rangle_{\delta t} \\ &= \langle k_i(t) \rangle_{\delta t} \times \langle g_i(t) \rangle_{\delta t}, \end{aligned} \quad (2)$$

$$\begin{aligned} s_i^2 &= \langle (k_i(t)g_i(t) - \langle k_i(t)g_i(t) \rangle_{\delta t})^2 \rangle_{\delta t} \\ &= \sigma_{k_i}^2 \langle g_i(t) \rangle_{\delta t}^2 - \langle k_i(t) \rangle_{\delta t}^2 \sigma_{g_i}^2 \end{aligned} \quad (3)$$

with σ_{k_i} , σ_{g_i} being the standard deviations of $k_i(t)$ and $g_i(t) \equiv g_i[\mathbf{x}(t)]$, respectively. Assuming now that the function g_i ’s variation in the time-window for which one evaluates the pair $m_i - s_i$ is negligible (i.e., $\sigma_{g_i}^2 \ll \sigma_{k_i}^2$), to a good approximation one has

$$\xi_i(t) = \frac{k_i(t) - \langle k_i(t) \rangle_{\delta t}}{\sigma_{k_i}}, \quad (4)$$

entailing that $\xi_i(t)$ has null average and unit standard deviation. If this assumption is correct the shape of the p_{Ξ} -distribution of the variable $\xi_i(t)$ should not depend upon $x_i(t)$. We have verified the hypothesis, *as our first result here*, with reference to all (8116) Spain’s municipalities. Fig. 1 displays the (x_i, ξ_i) -pairs for every township in the time-window $\delta t = 15$ years. From them we evaluate appropriate points taken at regular intervals from the cumulative distribution function of our random variable (quantiles) as a function of the population x . No apparent x -dependence can be detected. The overall distribution $p_{\Xi}(\xi)$ shape looks like a normal one

$$p_{\Xi}(\xi) = \frac{e^{-\xi^2/2}}{\sqrt{2\pi}}, \quad (5)$$

with cumulative distributions of the form

$$P_{\Xi}(\xi) = \frac{1}{2} \left[1 + \operatorname{erf} \left(\frac{\xi}{2} \right) \right], \quad (6)$$

shown in Fig. 1. Save for some fluctuations, we have not found any dependence on the shape of $p_{\Xi}(\xi)$ for the different provinces (same Fig. 1). Accordingly, but with a grain of salt, one may speak of “universality”. Consequently, we will consider herefrom that our variable ξ can be regarded as belonging to a Wiener process, *our second result here*.

IV. INTRODUCING MICROSCOPIC EQUATIONS OF MOTION

A. Proportional growth

We are now at step 3. For the g_i ’s shape we will assume that it depends only on its own x_i ’s population, i.e., $g_i[\mathbf{x}(t)] \simeq g_i[x_i(t)]$. In order to guess the explicit analytical form we appeal to a cluster-growth model in networks,^{8,17} used successfully before describing city-population distributions. We firstly consider a network of nodes (that eventually represents the social network) and a single node as seed of a cluster. Initially, the first neighbors of the seed will belong to the cluster with a given probability $P(t=0)$. At a subsequent time t , the first neighbors of the members of the cluster become also members with probability $P(t)$. Proceeding in this vein, it is reasonable to conjecture that the time-variation of the cluster-size \dot{x} at time t acquire the form

$$\dot{x}(t) = \sum_{j=1}^{x(t)} P(t) c_j(t), \quad (7)$$

where $c_j(t)$ is the first-neighbors-number of node i -th at time t . We appeal now to the *central limit theorem* to write

$$\dot{x}(t) = P(t) \left(\bar{c}(t)x(t) + \sigma_c(t)\sqrt{x(t)}\xi(t) \right). \quad (8)$$

Here $\bar{c}(t)$ is the mean neighbor’s number at time t , $\sigma_c(t)$ its standard deviation, and $\xi(t)$ an independent normally-distributed number. This last summand, usually neglected for very large sizes, is associated to finite-size effects. The first term, size-proportional, generates proportional (or multiplicative) growth. In view of this result, we consider the form

$$g_i(x_i) = [x_i]^\alpha \quad (9)$$

with $\alpha = 1$ or $1/2$. Considering then both terms in the microscopic dynamics we write

$$\dot{x}_i(t) = k_{i1}(t)x_i(t) + k_{i1/2}(t)\sqrt{x_i(t)}, \quad (10)$$

with the $k_{i1}(t)$ and $k_{i1/2}(t)$ two (a priori) independent Wiener coefficients. This dependence is checked out by

comparison of the previously employed s_i —numbers with a functional form of the type

$$\begin{aligned} s_i^2(x_i) &= \langle [\dot{x}_i]^2 \rangle_{\delta t} - \langle \dot{x}_i \rangle_{\delta t}^2 \\ &= \sigma_{i1}^2 x_i^2 + \sigma_{i1/2}^2 x_i, \end{aligned} \quad (11)$$

where σ_{i1} and $\sigma_{i1/2}$ are the associated deviations of $k_{i1/2}$ and k_{i1} , respectively. Rewriting (11) in a more convenient way we have

$$s_i^2(x_i)/x_i = \sigma_{i1}^2 x_i + \sigma_{i1/2}^2, \quad (12)$$

that, for sizes small enough reduces to

$$s_i^2(x_i)/x_i \approx \sigma_{i1/2}^2, \quad (13)$$

while for very large sizes one has

$$s_i^2(x_i)/x_i \approx \sigma_{i1}^2 x_i. \quad (14)$$

The transition between these two regimes should take place at a value $x_T = \sigma_{i1/2}^2/\sigma_{i1}^2$. Fig. 2 displays, as *our third result*, the $(x_i, s_i^2(x_i)/x_i)$ —pairs for all the Spanish municipalities, together with appropriate quantiles. The median $\operatorname{med}(s_i(x_i)/x_i)$ nicely fits things with $\sigma_{i1} = 0.0119$ and $\sigma_{i1/2} = 0.47$. We appreciate the fact that finite size fluctuations are larger than multiplicative ones, the later dominating, of course, for large sizes. Our transition occurs at population-values of the order of 1500 inhabitants. Surprisingly enough, the distribution of the variable $s'_i = \log[s_i(x_i)/\sqrt{x_i}] - \log[\operatorname{med}(s_i(x_i)/\sqrt{x_i})]$ becomes independent of x_i , being of a Gaussian nature.

At this point, we need still to address a further question. The finite-size term average is $\langle k_{i1/2}(t) \rangle_{\delta t} = 0$ (by definition), but this is not so for the multiplicative one $\langle k_{i1}(t) \rangle_{\delta t} \neq 0$, that is *a priori* regarded as constant and size-independent. This is indeed empirically true on occasions, but not always. For instance, such assumption cannot account for the migration from the countryside to big cities, where the mean growth rate correlates with the city-population.

B. Taking into account internal flow

It is a fact that small populations tend to diminish while large towns tend to increase their population. We encounter this scenario for most of the 50 provinces of Spain. We intend to tackle this issue below.

We can show that the effect can be described by recourse to a smooth dependence of the mean relative growth $\langle \dot{x}_i/x_i \rangle$ on $\log(\langle x \rangle)$ that generates what we will call internal flow. A second order expansion in $\log(\langle x \rangle)$ reads

$$\langle \dot{x}/x \rangle \simeq a + b \log(\langle x \rangle) + c \log(\langle x \rangle)^2 \quad (15)$$

where the values of a , b and c come from the corresponding Taylor coefficients. Assuming $b \gg c$ we can safely write it as

$$\langle \dot{x}/x \rangle \simeq \langle k_1 \rangle + \langle k_q \rangle [\langle x \rangle]^{q-1}, \quad (16)$$

where we have defined for convenience $\langle k_1 \rangle = a - b^2/2c$, $\langle k_q \rangle = b^2/2c$ and $q - 1 = 2c/b$. To validate our assumptions, we fitted the empirical provincial data to Eq. (16) via $\langle k_1 \rangle$, $\langle k_q \rangle$ and q , when possible (in some cases a quasi-linear relation is found, generating large uncertainty in the optimal values). We have found for the exponent q a mean value of 1.2 and a standard deviation of 0.45, with $|q - 1| < 1$ in all cases. This result confirms the assumption $b \gg c$ validating the second-order expansion of $\langle \dot{x}_i/x_i \rangle$. Moreover, as seen in Fig. 3 (our fourth result), nice fits are found in general with very few exceptions. With this new hypothesis our complete dynamic equation turns out to be

$$\dot{x}_i(t) = k_{iq}(t)[x_i(t)]^q + k_{i1}(t)x_i(t) + k_{i1/2}(t)\sqrt{x_i(t)}, \quad (17)$$

with $k_{iq}(t)$, $k_{i1}(t)$ and $k_{i1/2}(t)$ independent (a priori) Wiener processes. Summing up, we have assumed

- a finite size term that dominates things for low population levels (< 1500),
- a multiplicative term that accounts for population's growth/diminution (births, death or external migration, and
- a power-law (exponent $q \sim 1$) accounting for internal migration.

Since for most of the population range only one term dominates, we will include only one term in the considerations what follow below.

V. FROM MICROSCOPIC TO MACROSCOPIC DESCRIPTIONS

We arrive to stage 4, having discussed above a microscopic population dynamics. We will try now to ascertain whether a *macroscopic* description is also feasible. Our goal is to reduce the $2n$ microscopic degrees of freedom to a few macroscopic ones. We will separately consider each of the three terms of the dynamic equation. The ensuing results will be valid in the domains in which each term dominates.

Consider n random walkers characterized by a dynamic coordinate $x_i(t)$ obeying

$$\dot{x}_i(t) = k_i(t)[x_i(t)]^q, \quad (18)$$

with $\langle (k_i(t) - \bar{k})(k_j(t) - \bar{k}) \rangle = \sigma_k \delta_{ij} \delta(t - t')$. Parameter q will take as special possible values $1/2$ or 1 , or in general, $0 \leq q$.

A. Brownian motion and diffusion equation

We start with $q = 0$ as control case. One has $\dot{x}_i(t) = k_i(t)$ so that we deal with the well-known brownian random walkers. Consider this numerical procedure: initially, the

n walkers are located at, say, $x = x_0$. By $\rho(x, t)dx$ we will refer to the walker's normalized histogram, at time t , that indicates the walker's relative number positioned in the interval dx around x . The associated initial density would read $\rho(x, 0) = \delta(x - x_0)$. A discrete version of the pertinent dynamic equation is

$$x_i(t + \Delta t) = x_i(t) + \Delta t k_i(t), \quad (19)$$

that forces the walkers to “move” during the period Δt in a amount given by $\Delta t k_i(t)$, with $k_i(t)$ a random number generated from a Gaussian distribution determined by an standard deviation σ_k and mean \bar{k} , as defined above. We have

$$x_i(t = M\Delta t) = x_0 + \Delta t \sum_{m=1}^M k_i[(M-1)\Delta t], \quad (20)$$

so that after M iterations the walkers-distributions coincides with that of a random number generated by summing up M Gaussian numbers characterized by $\Delta t \sigma_k$ and $\Delta t \bar{k}$. Remind that a distribution that follows a random number composed of two other numbers of that character is the convolution of the distributions associated to these later numbers. Thus, $x(t)$ is described by the M -th convolution of the k 's Gaussian distribution. By recourse to a Fourier transform \mathcal{F} for convolutions we have

$$\begin{aligned} \mathcal{F}[\rho(x, t)] &= \left(\mathcal{F} \left[\frac{e^{-(k - \Delta t \bar{k})^2 / 2(\Delta t \sigma_k)^2}}{\sqrt{2\pi} \Delta t \sigma_k} \right] \right)^M \\ &= \left(e^{-\Delta t^2 \sigma_k^2 \omega^2 / 2 + i \Delta t \bar{k} \omega} \right)^M \\ &= e^{-M \Delta t^2 \sigma_k^2 \omega^2 / 2 + i M \Delta t \bar{k} \omega}, \end{aligned} \quad (21)$$

and, appealing to the inverse transformation,

$$\begin{aligned} \rho(x, t) &= \frac{e^{-(k - M \Delta t \bar{k})^2 / (2M(\Delta t \sigma_k)^2)}}{\sqrt{2\pi M \Delta t \sigma_k}} \\ &= \frac{e^{-(k - \bar{k})^2 / (4Dt)}}{\sqrt{4\pi Dt}}, \end{aligned} \quad (22)$$

where we have introduced for convenience $2D = \Delta t \sigma_k^2$. An arbitrary density $\rho(x, t)$ will evolve in Δt , via the convolution of that density with a Gaussian of deviation $\Delta t \sigma_k = \sqrt{2\Delta t D}$ and mean $\Delta t \bar{k}$, as

$$\begin{aligned} \mathcal{F}[\rho(x, t + \Delta t)] &= \mathcal{F}[\rho(x, t)] \times e^{-\Delta t D \omega^2 + i \Delta t \bar{k} \omega} \\ &\simeq \mathcal{F}[\rho(x, t)] (1 - \Delta t D \omega^2 + i \Delta t \bar{k} \omega), \end{aligned} \quad (23)$$

where we take Δt arbitrarily small. A simple manipulation involving division by Δt leads now to

$$\frac{\mathcal{F}[\rho(x, t + \Delta t)] - \mathcal{F}[\rho(x, t)]}{\Delta t} = (-D \omega^2 + i \bar{k} \omega) \mathcal{F}[\rho(x, t)]. \quad (24)$$

By recourse to the inverse transformation and taking the limit $\Delta t \rightarrow 0$ we get

$$\partial_t \rho(x, t) = D \partial_x^2 \rho(x, t) - \bar{k} \partial_x \rho(x, t), \quad (25)$$

which is a diffusion equation. Accordingly, we reach an important result here (our *fifth one*):

Our original $2n$ degrees of freedom-problem can now be tackled via just a few macroscopic parameters.

B. q -metric Brownian motion

In the general instance $q \neq 0$ we introduce a variable $u_i = \log_q(x_i)$, where \log_q is Tsallis' q -logarithm³⁸. The Jacobian for the transform is $du/dx = 1/x^q$ so that $\dot{u} = \dot{x}/x^q$ and the associated dynamical equation becomes

$$\dot{u}_i(t) = k_i(t). \quad (26)$$

In the set $\{(u_i, \dot{u}_i)\}_{i=1}^n$, the variables u_i and \dot{u}_i are independent of each other. We regard them, of course, as our dynamical variables. Note that one recovers Brownian motion for u . Indeed,

$$u_i(t = M\Delta t) = u_i(0) + \Delta t \sum_{m=1}^M k_i[(M-1)\Delta t], \quad (27)$$

and then the demonstration of the preceding subsection becomes valid, now for u and $\rho(u, t)du$. Our new diffusion equation reads

$$\partial_t \rho(u, t) = D \partial_u^2 \rho(u, t) - \bar{k} \partial_u \rho(u, t), \quad (28)$$

and, starting from a density $\rho(u, 0) = \delta(u - u_0)$ we end up with

$$\rho(u, t)du = \frac{du}{4\pi Dt} \exp \left[-\frac{(u - u_0 - \bar{k}t)^2}{4Dt} \right]. \quad (29)$$

The x -density is governed accordingly by a q -log-normal distribution

$$\begin{aligned} \rho_X(x, t)dx &= \rho[u(x), t] \frac{dx}{du} du \\ &= \frac{dx}{\sqrt{4\pi Dt} x^q} \exp \left[-\frac{(\log_q(x) - u_0 - \bar{k}t)^2}{4Dt} \right]. \end{aligned} \quad (30)$$

In particular, for $q = 1/2$ one has

$$\rho_X(x, t)dx = \frac{dx}{\sqrt{4\pi Dt} x} \exp \left[-\frac{(2(\sqrt{x} - 1) - u_0 - \bar{k}t)^2}{4Dt} \right], \quad (31)$$

and, for $q = 1$ the well known log-normal

$$\rho_X(x, t)dx = \frac{dx}{\sqrt{4\pi Dt} x} \exp \left[-\frac{(\log(x) - u_0 - \bar{k}t)^2}{4Dt} \right]. \quad (32)$$

We have again reduced the microscopic number of degrees of freedom to just a few macroscopic parameters.

C. Examples of diffusion

Numerical experiments confirm our findings above. We start with our dynamical equation in discrete form

$$x_i(t + \Delta t) = x_i(t) + \Delta t k_i(t) [x_i(t)]^q \quad (33)$$

using $k_i(t) = \sqrt{2D/\Delta t} \xi_i(t) + \bar{k}$, where the random numbers ξ follow a normal distribution such that $\langle \xi_i(t) \xi_j(t') \rangle = \delta_{ij} \delta(t - t')$. We have taken $q = 1/2$ and 1 for our examples, and find that the associated distributions exactly follow the diffusion equation's predictions. We have used in the former case $u_0 = \log_{1/2}(220)$, $\bar{k} = 0$, and $\sigma_k^2 = 10$, in intervals of $\Delta t = 0.01$. In the later instance we had $u_0 = \log(4400)$ instead. Indeed, the walkers' histograms' evolution follow Eq. (31) and Eq. (32), respectively, with $D = \Delta t \sigma_k^2 / 2$ as defined above (see Fig. 4 for the cumulative distributions).

As empirical examples we discovered that for small populations < 1500 inhabitants the finite-size noise dominates. Provinces for which most towns are scarcely populated will obey the dynamical equation with $q = 1/2$. Such is the case for the province of, i.e., Salamanca, as shown in top panel of Fig. 4. The ensuing dynamics confirms this assertion. The relative growth of most of the towns follows a dynamics with a variance $s^2 \propto \sqrt{x}$ (red line of the inset). The ensuing distribution fits the final state predicted by the diffusion equation for that dynamics, Eq. (31), with $u_0 + \bar{k}t = \log_{1/2}(216.3)$ and $2Dt = 95.6$ for year 2010 (see Fig. 4). Remark that the $1/2$ -log-normal can be easily confused with the usual log-normal, although the former exhibits asymmetries in log-scale. As a $q = 1$ -example we mention Florida State in the US³⁹ (see also bottom Fig. 4). Using data from 1990, 2000, and 2010, we have verified that the microscopic dynamics confirms the proportional growth assumption (with a variance of the relative growth independent of the size, as illustrated in the inset). The city-populations distribution follows a log-normal distribution, that of Eq. (32), which can be the one pertaining to geometrical random-walkers' diffusion, with $u_0 + \bar{k}t = \log(4380)$ and $2Dt = 2.96$.

D. Constrained diffusion

q -log-normal distributions do not set any limits to population-sizes. However, it is reasonable to assume that physical space does pose limits to a city's population-growth. Unlimited growth is unrealistic since in the case of internal migrations the total population N_T should remain constant and a free-diffusion model is, again, unrealistic. Constrained diffusion must be contemplated instead.

We pass now to consider numerical experiments with random walkers that fix lower and upper bounds for population. These are denoted by x_0 and x_M , respectively.

Now, walkers “moves” leading to values outside the range $x_0 < x < x_M$ are to be rejected in our simulations. Fig. 5 shows that a q -metric walkers’ evolution begins by faithfully following the diffusion equation Eq. (28) till they bump off these extreme values. Now their density deviates from that of “free” evolution. After some time has elapsed, an equilibrium x -distribution is reached that follows a power-law with exponent q , independently of the initial state. *The origin of this systematic result can not be unraveled by the simulations, so a higher-level of theory is needed.*

Now we use a total population constraint. This is equivalent to make the walkers move under the rule of a q -generalized multi-component logistic equation

$$\dot{x}_i(t) = [x_i(t)]^q \left[k_i(t) - \frac{\sum_{i=1}^n k_i(t) [x_i(t)]^q}{\sum_{i=1}^n [x_i(t)]^q} \right]. \quad (34)$$

Indeed, it is easy to check that $\partial_t N_T = \sum_{i=0}^n \dot{x}_i(t) = 0$, thus preserving the value of N_T in time. Also the original q -symmetry of the dynamics is preserved. This equation is the q -generalization of the scale-invariant multi-component logistic equation presented in 35. Results are displayed in Fig. 6 for $q = 1, 1.5$ and 2 , using $n = 100000$ walkers and a total population of $N = 250000$ inhabitants (with $x_0 = 1$). Remarkably enough, equilibrium is always reached, to a density that does not depend upon the initial state or the k -parameters. The shape of the distributions resembles x power-laws with exponential cut-off. Again, the simulation can not unravel the origin of this form. *Finding the properties and the exact analytical form of those macroscopic equilibrium distributions is our goal in the next Section.*

VI. THE MACROSCOPIC CONUNDRUM

We tread now step 5. Our simulations with random walkers suggest that it is indeed possible to pass from a description that uses $2n$ microscopic variables to a description involving just a few macroscopic parameters. The big question is: do they behave in thermodynamic fashion, satisfying the pertinent partial derivatives-relationships? We wish to tackle this issue now looking for a way to reduce the number of microscopic degrees of freedom to a few manageable macroscopic ones while keeping a coherent, reasonable description of our system, mimicking the kind of scenario that links statistical mechanics to thermodynamics. This requires appropriate constraints, a topic to be addressed below by enumerating the appropriate “social” constraints we need.

A. Macroscopic constraints

- *Total number of cities n .* Since there is some confusion in the available data about what the administrative

meaning of city is, we wish to ascertain that this issue is of no importance. Consider $x_i = \sum_j^{n_i} x_{ij}$, where n_i is the number of sub-administrative units included in the administrative unit i , with x_{ij} their sub-administrative populations. Considering proportional growth, we write for the time-evolution

$$\begin{aligned} \dot{x}_i(t) &= \sum_j^{n_i} \dot{x}_{ij}(t) \\ &= \sum_j^{n_i} k_{ij}(t) x_{ij}(t) \\ &= \frac{\sum_j^{n_i} k_{ij}(t) x_{ij}(t)}{\sum_j^{n_i} x_{ij}(t)} \sum_j^{n_i} x_{ij}(t) \\ &= k'_i(t) x_i(t), \end{aligned} \quad (35)$$

where we have defined $k'_i(t)$ as a new variable defined as an average weighted by the populations x_{ij} . If the growth rates k_{ij} are random variables with approximately the same mean and variance, it is easy to check that $k'_i(t)$ is in turn a random variable of the same mean and variance. The dynamical behavior of the ensemble of administrative units \mathbf{x} is thus equivalent of that of the sub-units, and the procedure described in this work is still applicable.

- *Maximum/minimum population x_M/x_0 .* It is well-known that a typical minimum population size equals the Dunbar number⁴⁰ (~ 150), heuristically associated to the maximum (allowable by our neo-cortex) number of stable human relationships. Thus, it is reasonable to think of a minimum size $x_0 \sim 150$. In many cases a maximum number for a city population x_M can be established via consideration of geographical peculiarities as mountains⁴¹ or oceans⁴² (See Fig. 7 for an example). In such cases it is convenient to employ the transform $u = \log_q(x/x_0)$. An associated, valuable macroscopic parameter is $u_M = \log_q(x_M/x_0)$. We will be dealing then with a “volume” $0 < u < u_M$.

- *Total population N_T .* We have $N_T = \sum_{i=1}^n x_i$ that gets transformed into $N_T = x_0 \sum_{i=1}^n e_q^{u_i}$. A useful quantity becomes then $N = N_T/x_0$.

- *Total variance of \dot{u} .* With reference to the dynamics, a useful observable is the total variance for relative growth $\sigma^2 = \sum_{i=1}^n \langle (\dot{u}_i - \langle \dot{u}_i \rangle_t)^2 \rangle / n$. For a Gaussian form (see Fig. 1) this quantity measures fluctuation-intensities. Generalizing, this quantity can be defined by the covariance matrix with elements $Q_{ij} = \langle (\dot{u}_i - \langle \dot{u}_i \rangle)(\dot{u}_j - \langle \dot{u}_j \rangle) \rangle$, using its trace as a thermodynamical variable $\text{Tr}(Q) =$

$$\sum_{i=1}^n Q_{ii}:$$

$$\begin{aligned} U &= \frac{\tau}{2} \text{Tr}(Q) \\ &= \frac{\tau}{2} \sum_{i=1}^n \langle (\dot{u}_i - \langle \dot{u}_i \rangle)^2 \rangle \\ &= \frac{\tau}{2} n \sigma^2, \end{aligned} \quad (36)$$

where we add for dimensional convenience a factor $\tau/2$.

B. Fundamental hypothesis for urban-thermodynamics

Let us discuss the pertinent three hypothesis that we need in our Scheme:

- *H-I. Microscopic hypothesis.*

We adopt as fundamental dynamical equation Eq. (18) [$\dot{x} = kx^q$] for the population of a center, linearized via the variable $u = \log_q(x/x_0)$. We will think of the pair (u, \dot{u}) as constituting our social phase space coordinates. We can speak of an:

- *H-II. A priori phase space equiprobability in (u, \dot{u}) .*³⁷

The probability density distribution for the i -th phase space cell centered at (u_i, \dot{u}_i) of size $dud\dot{u}$ is defined as $\rho[\{(u_i, \dot{u}_i)\}_{i=1}^n] d^n u d^n \dot{u}$. Accordingly to H-II, the system's entropy is written as

$$S[\rho] = - \int d^n u d^n \dot{u} \rho[\{(u_i, \dot{u}_i)\}_{i=1}^n] \log [\rho[\{(u_i, \dot{u}_i)\}_{i=1}^n]]. \quad (37)$$

Since none of our macroscopic observables is able to distinguish amongst population nuclei, towns are thus indistinguishable. In this case, the useful distribution is the one-body density $\rho(u, \dot{u})$ defined as

$$\rho(u, \dot{u}) = \int d^{n-1} u d^{n-1} \dot{u} \rho[\{(u_i, \dot{u}_i)\}_{i=1}^n], \quad (38)$$

and thus,

$$S[\rho] = - \int dud\dot{u} \rho(u, \dot{u}) \log [\rho(u, \dot{u})]. \quad (39)$$

Macroscopic observables are written in terms of the one-body density as

$$n = \int dud\dot{u} \rho(u, \dot{u}), \quad (40)$$

$$N = \int dud\dot{u} \rho(u, \dot{u}) e_q(u), \quad (41)$$

$$U = \frac{\tau}{2} \int dud\dot{u} \rho(u, \dot{u}) \dot{u}^2. \quad (42)$$

- *H-III. Maximum entropy principle (MaxEnt).*³⁷

Equilibrium is determined via constrained entropic maximization using n , u_M , N y U . This determines the equilibrium density $\rho(u, \dot{u})$ that is a solution of the entropic variational problem

$$\delta \{S[\rho] - \beta A[\rho]\} = 0, \quad (43)$$

with

$$A = U - \mu n + p u_M + \Lambda N, \quad (44)$$

where β , μ , p and Λ stand for the pertinent Lagrange multipliers, that will be seen below to acquire the character of intensive thermal-quantities.

C. Thermodynamical relations

We enter step 6 by considering the *Lagrangian* $A[\rho]$ [and Lagrangian density $a(u, \dot{u})$]. It reads

$$\begin{aligned} A[\rho] &= \int dud\dot{u} \rho(u, \dot{u}) a(u, \dot{u}) \\ &= \int dud\dot{u} \rho(u, \dot{u}) \left\{ \frac{\tau}{2} \dot{u}^2 - \mu + p v(u) + \Lambda e_q(u) \right\}, \end{aligned} \quad (45)$$

where the volume condition is enforced by an infinite-well potential

$$v(u) = \begin{cases} u_M/n & \text{for } 0 < u < u_M; \\ \infty & \text{otherwise} \end{cases} \quad (46)$$

The well-known general solution to the entropic problem Eq. (43) is³⁷ $\rho(u, \dot{u}) = \exp[-\beta a(u, \dot{u})]$, so that

$$\rho(u, \dot{u}) = \frac{n}{Z} e^{-\frac{\beta\tau}{2} \dot{u}^2 - \beta \Lambda e_q(u)} \quad (0 < u < u_M), \quad (47)$$

where the normalization factor Z (partition function) becomes

$$\begin{aligned} Z &= \int_{-\infty}^{\infty} d\dot{u} \int_0^{u_M} du e^{-\frac{\beta\tau}{2} \dot{u}^2 - \beta \Lambda e_q(u)} \\ &= \sqrt{\frac{2\pi}{\beta\tau}} E_q(\beta\Lambda, u_M), \end{aligned} \quad (48)$$

with $E_q(l, m)$ the generalized exponential function of order q

$$E_q(l, m) = E_q(l) - e^{(1-q)m} E_q(l e^m). \quad (49)$$

Our constraints in U and N determine the multipliers β and Λ -values. On the one hand, we have the \dot{u} -variance

$$\begin{aligned} U &= \frac{\tau}{2} n \int_{-\infty}^{\infty} d\dot{u} \sqrt{\frac{\beta\tau}{2\pi}} e^{-\frac{\beta\tau}{2} \dot{u}^2} \dot{u}^2 \\ &= \frac{n}{2\beta}, \end{aligned} \quad (50)$$

and on the other one, we have to deal with the total population

$$\begin{aligned} N &= n \int_0^{u_M} du \frac{e_q(u) e^{-\beta \Lambda e_q(u)}}{E_q(\beta \Lambda, u_M)} \\ &= n \frac{E_{q-1}(\beta \Lambda, u_M)}{E_q(\beta \Lambda, u_M)}, \\ &= n F_q(\beta \Lambda, u_M), \end{aligned} \quad (51)$$

where we use the function $F_q(l, m) = \partial_l \log[E_q(l, m)]$. We obtain from the former the direct result

$$\beta = \frac{n}{2U}, \quad (52)$$

and, via inversion of the latter equation [defining first $L_q(f, m) = F_q^{-1}(f, m)$ and thus $F_q[L_q(f, m), m] = f$], we finally obtain the relation between the system variables (*equation of state*)

$$\beta \Lambda = L_q(N/n, u_M). \quad (53)$$

Note that we have intensive quantities on the left hand side, while extensive ones appear in the r.h.s.. The entropy becomes

$$\begin{aligned} S &= n \log \left[\frac{1}{n} \sqrt{\frac{2\pi}{\beta\tau}} E_q(\beta \Lambda, u_M) \right] \\ &\quad + n \left[\frac{1}{2} + \beta \Lambda F_q(\beta \Lambda, u_M) \right]. \end{aligned} \quad (54)$$

Using now Eq. (53) we can recast things in term of the natural variables as

$$\begin{aligned} S(U, n, u_M, N) &= n \log \left[\frac{2}{n} \sqrt{\frac{\pi U}{n\tau}} E_q[L_q(N/n, u_M), u_M] \right] \\ &\quad + \frac{n}{2} + L_q(N/n, u_M) N. \end{aligned} \quad (55)$$

It is easy to verify, but crucial to our present goals, that macroscopic observables and Lagrange multipliers become linked entropic-wise via

$$\beta = \left. \frac{\partial S}{\partial U} \right|_{n, u_M, N}, \quad (56)$$

$$\mu = \frac{1}{\beta} \left. \frac{\partial S}{\partial n} \right|_{U, u_M, N}, \quad (57)$$

$$p = \frac{1}{\beta} \left. \frac{\partial S}{\partial u_M} \right|_{U, n, N}, \quad (58)$$

$$\Lambda = \frac{1}{\beta} \left. \frac{\partial S}{\partial N} \right|_{U, u_M, n}. \quad (59)$$

The first relation leads to Eq. (52), also showing that the β -multiplier is the inverse of the variance $\beta = 1/\tau\sigma^2$. The last relation takes us to Eq. (53) and is indeed one of our equations of state. The other two are

$$\begin{aligned} \beta\mu &= \log \left[\frac{2}{n} \sqrt{\frac{\pi U}{n\tau}} E_q[L_q(N/n, u_M), u_M] \right] - 1 \\ &\quad - 2L_q^{(1,0)}(N/n, u_M) \left(\frac{N}{n} \right)^2 \end{aligned} \quad (60)$$

and

$$\begin{aligned} \beta p &= n \frac{\exp[-L_q(N/n, u_M) e_q(u_M)]}{E_q[L_q(N/n, u_M), u_M]} \\ &\quad + 2L_q^{(0,1)}(N/n, u_M) N. \end{aligned} \quad (61)$$

At this stage the reader will agree that it is fair to assert that our goal has been successfully reached. *We have indeed constructed a social thermodynamics for urban population flows.* The following equivalence may be established vis-a vis the thermodynamics of chemical species:

- Temperature \leftrightarrow Variance of relative growth.
- Number of inhabitants \leftrightarrow Number of particles.
- Number of towns \leftrightarrow Number of chemical species.
- Volume \leftrightarrow Maximum possible town's population.

VII. APPLICATION: THE SCALE-FREE IDEAL GAS (SFIG)

This is our final step 7. We envision two main regimes, according to the Λ -value: $\Lambda = 0$ and $\Lambda > 0$.

A. The SFIG-in-a-box

We will consider in some detail the first case here. Different scenarios can be associated to $\Lambda \rightarrow 0$: i) the system is not isolated and exchanges population with its surroundings, with a maximum-size constraint, ii) the triplet n, N, u_M is such that the equation of state yields $\Lambda = 0$, i.e., $N/n = \log_{q-1}[e_q(u_M)]/u_M$, or iii) no size-limitation exists ($u_M \rightarrow \infty$) but N/n is large enough to consider $\Lambda \sim 0$. In the latter case one can obtain an effective u_M -value from normalization such that $u_M = E_q(\beta \Lambda)$. When $\Lambda = 0$ the Lagrangian A is written as

$$A = U - \mu n + p u_M, \quad (62)$$

so that we do not need knowledge of N . The equilibrium density is

$$\rho(u, \dot{u}) du d\dot{u} = \frac{n}{u_M} \sqrt{\frac{\beta\tau}{2\pi}} e^{-\frac{\beta\tau}{2} \dot{u}^2} du d\dot{u} \quad (0 < u < u_M). \quad (63)$$

The partial density $\rho(u) = \int d\dot{u} \rho(u, \dot{u}) = n/u_M$ is constant in u so that x is given by a power-law

$$\rho_X(x)dx = \rho[u(x)] \frac{du}{dx} dx = n \frac{x_0^{q-1}}{u_M} \frac{dx}{x^q}, \quad (64)$$

with an associated rank-plot given by

$$x(r) = x_0 e_q[u_M(1 - r/n)], \quad (65)$$

where r is the rank from 1 to n . Comparing with the equilibrium densities found above in our numerical experiments with random walkers, a very nice fit ensues as seen in Fig. 5, which *validates our methodology*. The entropy becomes

$$S(U, n, u_M) = n \log \left[\frac{2}{n} \sqrt{\frac{\pi U}{n\tau}} u_M \right] + \frac{n}{2}, \quad (66)$$

resembling that of the one-dimensional ideal gas. The state-equations are

$$\beta\mu = \log \left[\frac{2}{n} \sqrt{\frac{\pi U}{n\tau}} u_M \right] - 1, \quad (67)$$

and

$$\beta p = \frac{n}{u_M}, \quad (68)$$

in exact agreement with the ideal gas scenario.

As empirical $q = 1$ -examples we show the cases of i) Marshall Islands,⁴³ ii) d'Agosta Valley (Italy)⁴⁴ and iii) Huelva-province (Spain)³⁶. In all instances the relative growth is nearly independent of the population (with some finite-size noise) as in the case of Huelva, or with a secondary constant trend for low-populated cities, as in d'Agosta Valley's instance. Thus we consider that the microscopic dynamics fits the proportional growth hypothesis, with densities $\rho(u, \dot{u})$ nicely adapted to the ensuing thermodynamic predictions. In all cases, geographical conditions set strong limits to the city-sizes. The values for the macroscopic parameters are shown in Table I. Remarkably enough, the pressure p due to the limited space is highest for the Marshall Islands. Indeed, this system exhibits the lowest volume u_M and the lowest β (highest "temperature") for a large number of units n .

	Marshall Islands	Agosta Valley	Huelva
n	160	74	79
x_0	2.14	126	206
u_M	0.038	0.048	0.059
β	0.504	2.05	11.7
p	0.829	0.0133	0.0115

TABLE I: Macroscopic parameters for the four SFIG-in-a-box examples ($\tau = 100$ years²).

B. The SFIG under total population-constraint

We now consider $\Lambda > 0$, with $u_M \rightarrow \infty$ for simplicity. This case describes regions where internal migration dominates the microscopic dynamics, and no upper limit is found for the city-size. According to the equation of state Eq. (61), $p = 0$ in this limit, so that we deal with the lagrangian

$$A = U - \mu n + \Lambda N. \quad (69)$$

The equilibrium density is

$$\rho(u, \dot{u}) du d\dot{u} = \frac{n}{E_q(\beta\Lambda)} \sqrt{\frac{\beta\tau}{2\pi}} e^{-\frac{\beta\tau}{2}\dot{u}^2 - \beta\Lambda e_q(u)} du d\dot{u} \quad (0 < u), \quad (70)$$

and the equation of state can be written in the form

$$N = n \frac{E_{q-1}(\beta\Lambda)}{E_q(\beta\Lambda)}. \quad (71)$$

The partial density for x is given by a power-law with exponential cut-off

$$\rho_X[x]dx = n \frac{x_0^{q-1}}{E_q(\beta\Lambda)} \frac{e^{-\beta\Lambda x}}{x^q} dx, \quad (72)$$

with an associated rank-plot

$$x(r) = \frac{x_0}{\Lambda} E_q^{-1}[E_q(\Lambda)r/n]. \quad (73)$$

Again, this result fits the numerical equilibrium densities found above in our numerical simulations (Fig. 6), *validating again our methodology*.

This is the most common situation in the Spanish provinces (for more details see Ref.³³). We have found nice agreement between the q value obtained from a fit to the microscopic dynamics and the q -value obtained from the fit of the rank-plot to Eq. (73). We show some examples in Fig. 9, and the associated macroscopic numerical results in Table II. In the examples presented below, using the parameter Λ as a measure of the pressure generated by the total population constraint, it turns out that Alicante is the province with the highest pressure and Girona that with the lowest one, correlated with a highest and a lowest "temperature", respectively.

VIII. CONCLUSIONS

After initially introducing some useful social-macroscopic and social-stochastic quantities we have

1. Postulated social, dynamic microscopic equations.
2. Validated them using urban population data.
3. Performed numerical simulations with random walkers that conclusively demonstrated that a description using many microscopic variables has as a counterpart a macroscopic one with few parameters.

	Alicante	Almería	Girona	Lleida
n	140	101	220	230
x_0	83.9	147	141	105
N	18355.7	3245.53	4597.37	2818.12
$\log(\beta\Lambda)$	-6.26	-5.35	-5.18	-4.02
Λ	$4.03 \cdot 10^{-3}$	$1.83 \cdot 10^{-4}$	$7.89 \cdot 10^{-5}$	$4.1 \cdot 10^{-4}$
β	0.47	25.9	71.1	43.8
q	0.862	1.135	1.27	1.16
	Navarra	Vizcaya	Zaragoza	Granada
n	271	111	292	167
x_0	47.6	219	40.7	263
N	8976.55	3571.01	6906.03	2539.5
$\log(\beta\Lambda)$	-5.16	-5.18	-5.06	-4.01
Λ	$6.69 \cdot 10^{-4}$	$1.18 \cdot 10^{-4}$	$6.22 \cdot 10^{-4}$	$2.03 \cdot 10^{-3}$
β	8.57	47.4	10.2	8.91
q	1.06	1.08	1.18	1.02

TABLE II: Macroscopic parameters for the SFIG-under-pop-constraint examples.

4. Showed that such macroscopic description can be given an appropriate MaxEnt form after constructing a “social” phase space, that allows one to derive thermodynamic-like relations amongst our macro-parameters.

5. Finally, as an application, we successfully analyzed urban flows as modelled by a scale invariant ideal gas.

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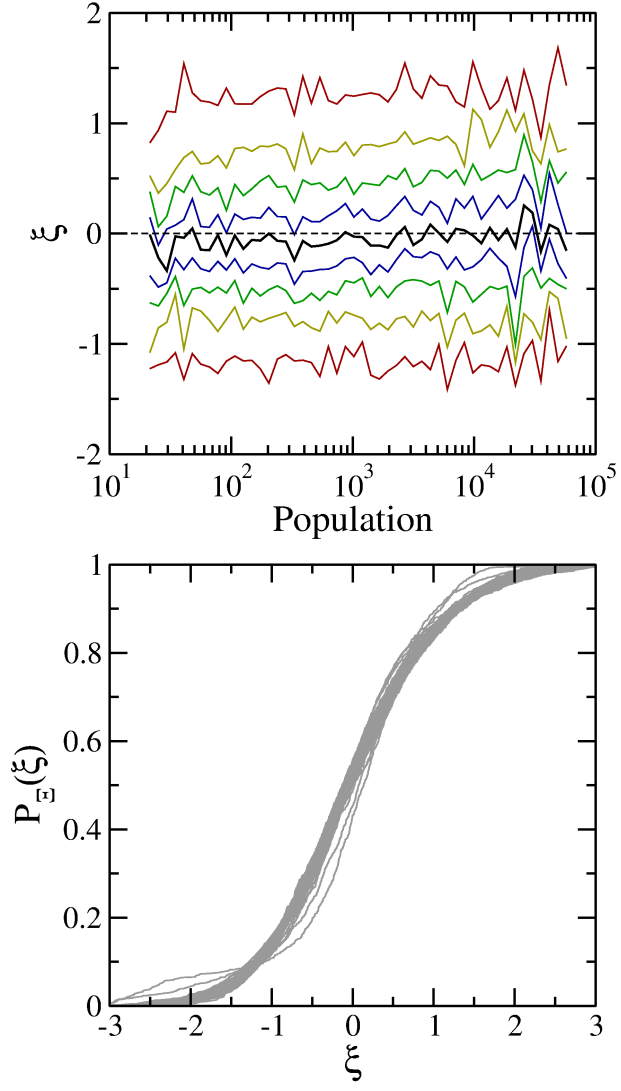


FIG. 1: Top panel: quantiles from 0.1 to 0.9 each 0.1 for $p_{\Xi}(\xi)$ as a function of the population x (the median is shown in black). Bottom panel: $P_{\Xi}(\xi)$'s cumulative distribution for each of Spain's provinces

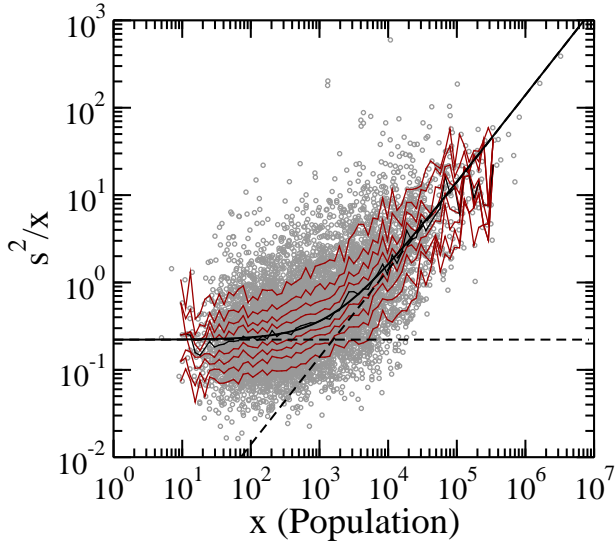


FIG. 2: Variance s^2/x vs. x . Red: quantiles from 0.1 to 0.9 each 0.1. Solid black: fit to the median value. Dashed black lines: Finite-size's fluctuations are constant, while the multiplicative regime is given by a straight line.

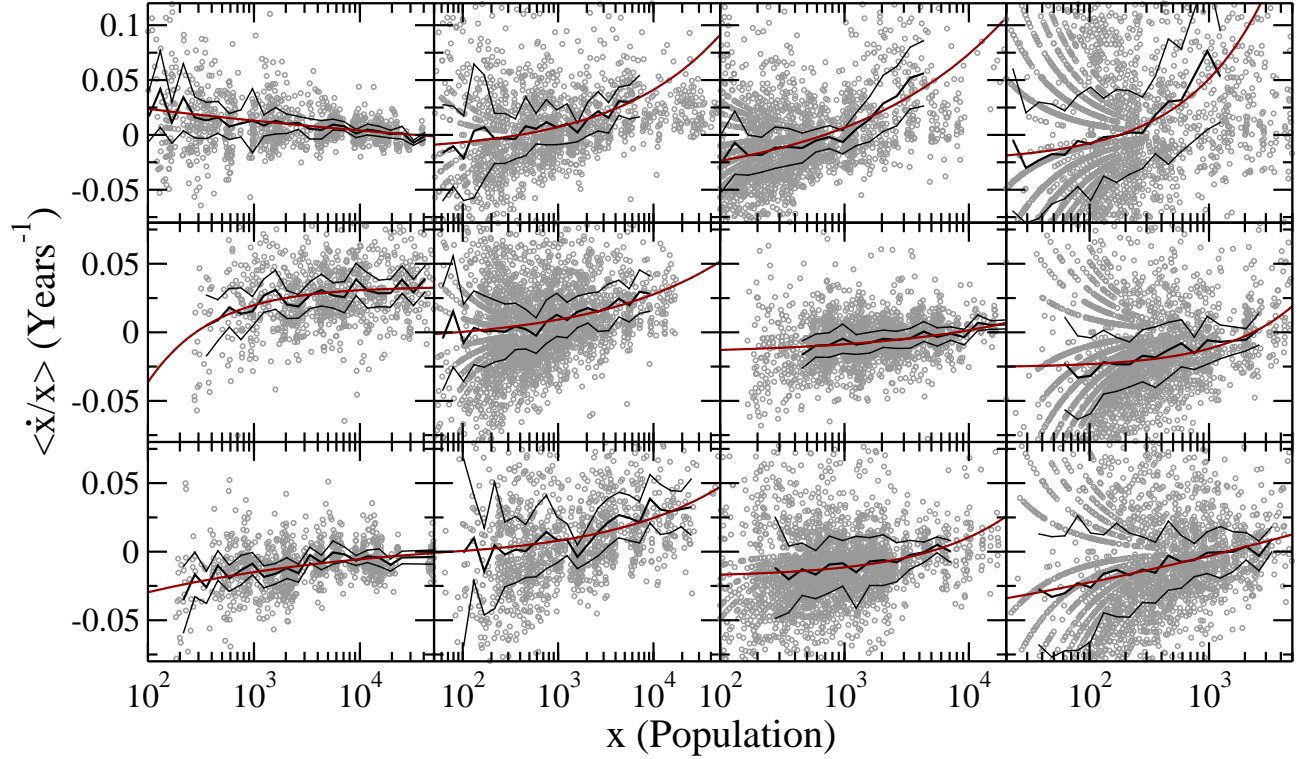


FIG. 3: Fit of $\langle \dot{x}/x \rangle$ to an expression of the type represented by Eq. (16) for 12 Spanish provinces: Asturias, Almería, Cáceres, Cuenca, Baleares, Lleida, Badajoz, Ávila, Guipúzcoa, Castellón, Valladolid and Guadalajara.

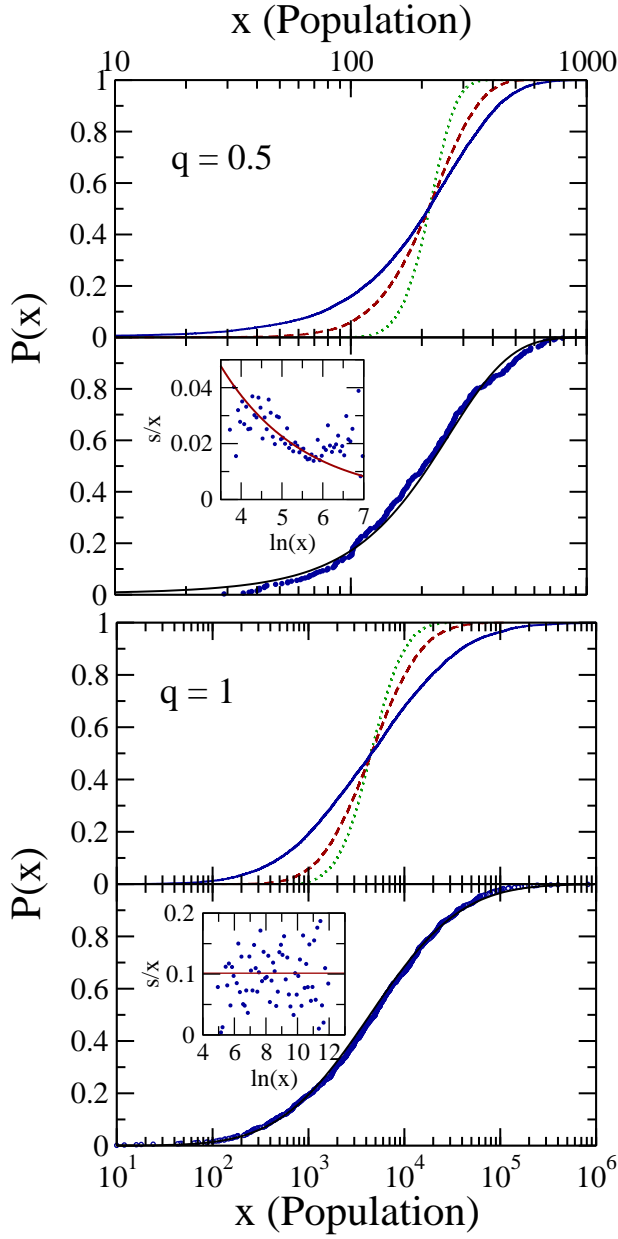


FIG. 4: Top panel: $q = 21/2$ -metric diffusion at times $t = 20$ (green dotted line), 70 (red dashed) and 110 (blue solid) using the text-parameters compared with the distribution of Salamanca towns' population in 2010, fitted to a $1/2$ -log-normal distribution (solid line). Inset: variance of the relative growth vs. log-population (dots), confirming the \sqrt{x} dependence for $q = 1/2$ dynamics (red line). Bottom panel: geometric diffusion ($q = 1$) at times $t = 4$ (green dotted), 9 (red dashed) and 29 (blue solid) compared with the population distribution of Florida State (US) in 2010, fitted to a log-normal distribution (solid line). Inset: same as top panel's inset, confirming size independence and thus proportional dynamics.

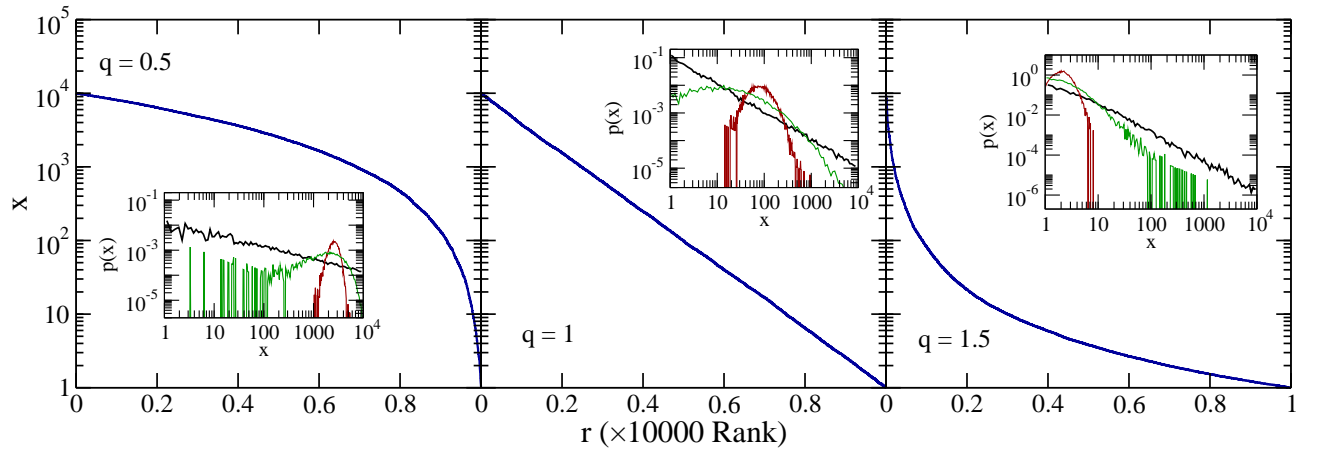


FIG. 5: q -metric diffusion with maximum size constraint for (from left to right panels) $q = 1/2$, 1 and 1.5. Rank-distributions and evolution (insets).

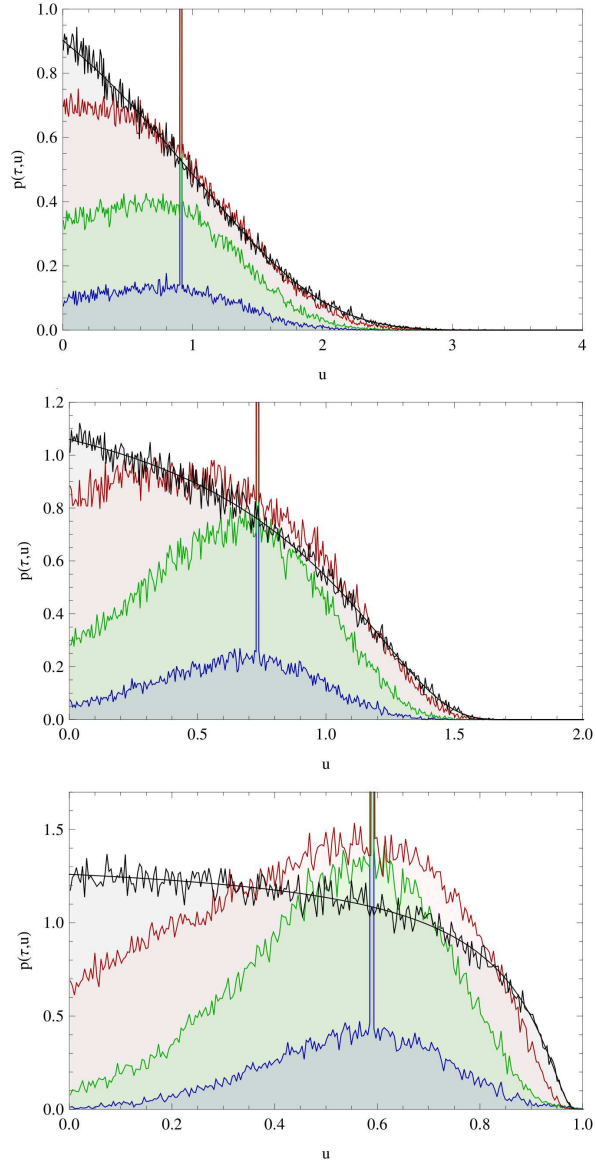


FIG. 6: N -constrained q -metric diffusion for $q = 1, 1.5$, and 2.



FIG. 7: An example of population restriction arising out of geographical reasons. Left Panel: d'Aosta valley (Italy)⁴¹. Right Panel: Marshall islands⁴².

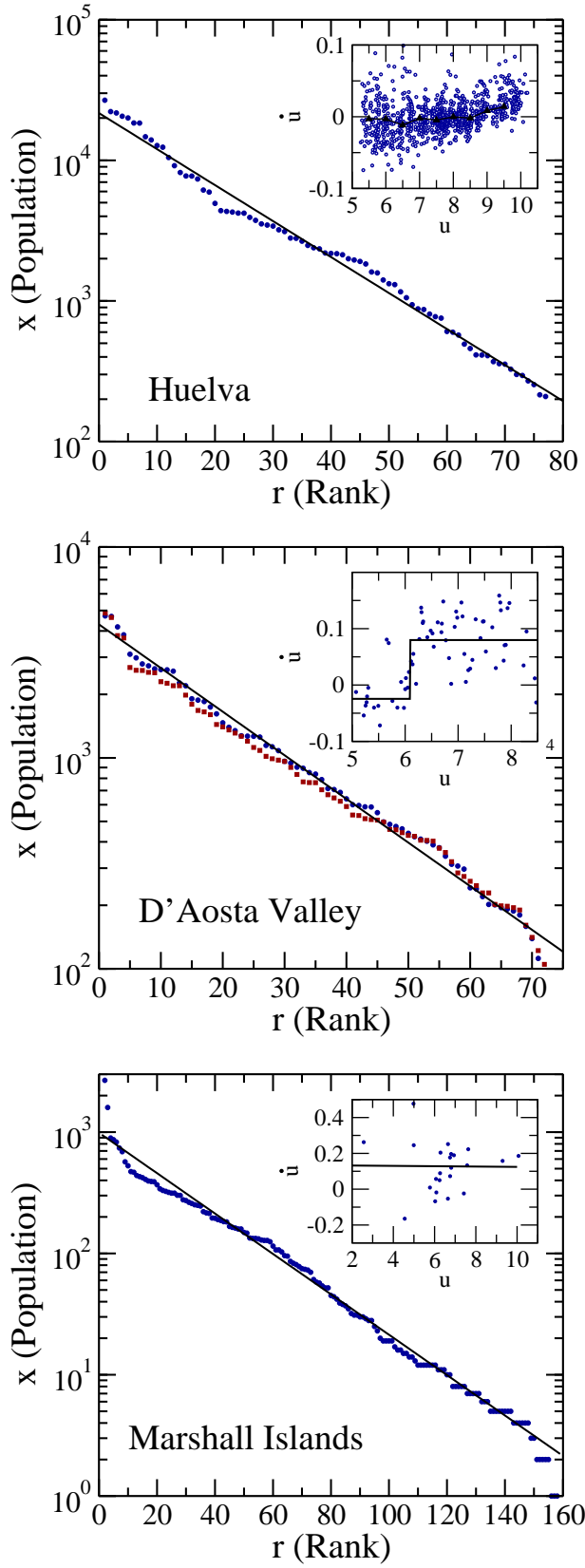


FIG. 8: SFIG-in-a-box examples, from top to bottom: RD of Huelva-province (Spain), D'Aosta Valley (Italy), and Marshall Islands (dots), compared with distribution Eq. (65) (lines). In the insets, the relative growth \dot{u} vs. the logarithmic population u .

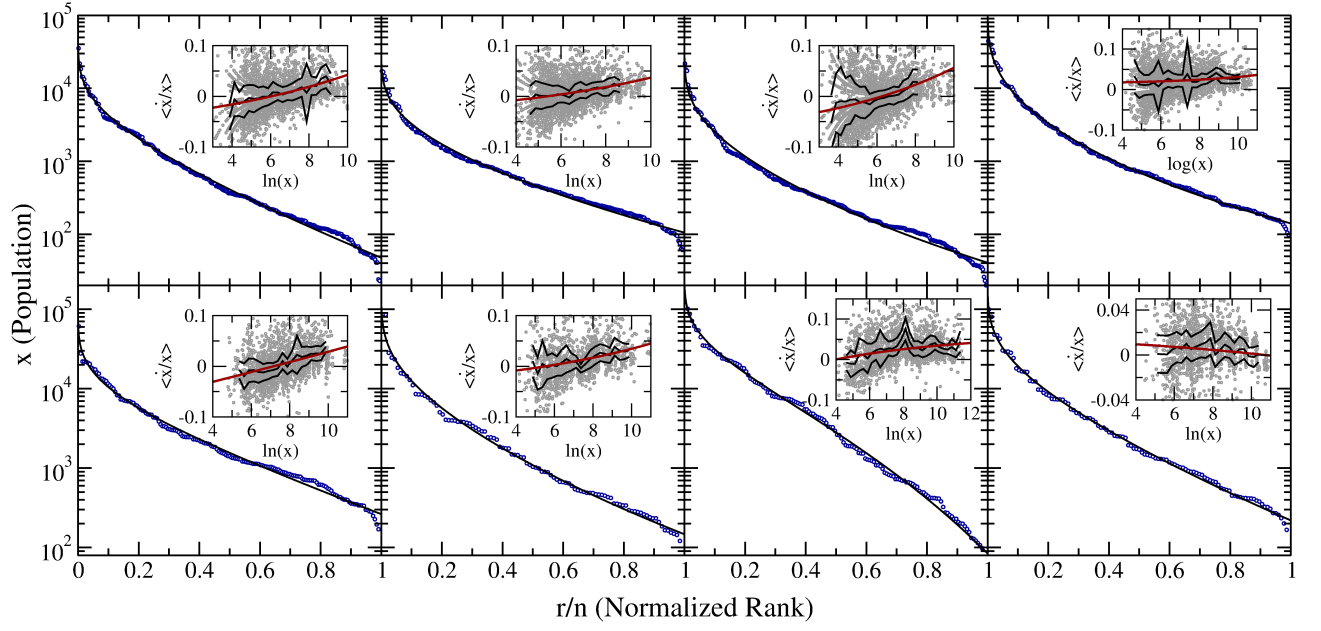


FIG. 9: SFIG under total-population constraint examples, in the reading order: RD of Navarra, Lleida, Zaragoza, Girona, Granada, Almería, Alicante, and Vizcaya (dots) and RD of Eq. (73) (lines). Insets: relative growth \dot{x}/x vs. log-population $\log x$ fitted to Eq. (16) with the same value of q as in the RD.